The Number Concept: Human Cognition and Philosophy of Mathematics

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1 Motivations

The general objective of this paper is to contribute to the understanding of the *concept of natural number*.

Different disciplines study "natural numbers" under many different incomparable angles: in mathematics as abstract entities, in computer sciences, as certain class of inscriptions, in linguistics as quantifiers, in psychology as mental representations, in neuro-psychology as neuron configurations, in philosophy as Platonic abstract objects, etc. This list is not exhaustive, moreover, each of the disciplines can study numbers under many different aspects: ex. in mathematics natural numbers can be understood in set theoretical terms (ex. to be identified with finite von Neumann's ordinals) or, studied from the axiomatic viewpoint, as these abstract entities which are described by the second-order Peano Arithmetic. However, there is no general consensus of what natural numbers are.

Even though one can find exchanges and influences between some of the listed disciplines (ex. certain schools in philosophy of mathematics aim to give an account of actual mathematical practice), more often each of these disciplines develops in isolation from the influence of the others, using exclusively its own formal or experimental tools. The debates are eventually conducted within one field (philosophers with various ontological orientations discuss with each other, developmental psychologists disagree on certain aspects of how exactly the number concept is constructed in infants), but there is little inter- or multidisciplinary exchange on what natural numbers are, and how do we know what they are.

The specific objective of this paper is to investigated the possible connections (relations, interactions and boarder line) between two of those different approaches to study of the number concept: the *philosophy of mathematics* (related to study of natural numbers) and the *cognitive sciences* oriented towards number cognition.

2 Background assumptions

The background assumption is that there is a common object of studies and that both disciplines can improve its own achievements by taking inspiration in the other fields' results and methodology. This assumption is modulated as follows:

- there are various aspects of numerical concepts, especially in the early stage of individual development;
- different branches of the philosophy of mathematics highlight different, not necessarily incompatible aspects of the number concept.

3 Objective

The problem, which will be discussed in the proposed talk, is devoted to an attempt of showing how several philosophical and mathematical intuitions concerning foundations of arithmetic can be reconciled. In particular, we focus attempts to reconcile the following intuitions:

- the intuition that natural numbers serve for counting and computing (brought out by cognitive scientists),
- the intuition that natural numbers are amenable to treatment as a mathematical structure in the sense of model-theory (observed by philosophers of mathematics).

The discrepancy between these two intuitions has been recently highlighted by *computational structuralism*. A model of arithmetic which is claimed to be intended, according to this position, is a recursive (or computable) model of arithmetic, and the identity between models is claimed to be established by some *computable* isomorphism. In consequence, intended models form a proper subclass of standard models (ω-ordered models of arithmetic), and an identifying functions are a subclass of all isomorphisms, identifying standard models. Accepting the standpoint of computational structuralists results in rejecting as intended the class of non-computable standard models and is incompatible (at the first glance) with intuitions of mathematicians working in model-theoretical framework.

4 Argumentation Line

A conceptual analysis proposed in this paper, takes into account various stages of number concept formation. It starts by studying the research of cognitive scientists, and extends to the objectives of philosophers of mathematics. In particular it explores a conceptual possibility of founding the concept of natural numbers of mathematicians (which is called here "saturated", and is opposed to “open-textured” concepts) on the intuitive concept of computability (understood as issued from the innate cognitive number systems).

The proposed picture is three-folded:

1. initial cognitive numerical systems,
2. computability intuitions,
3. formal definitions.

4.1 Stage one: innate cognitive systems and informal intuitions

The first fold corresponds to the *innate cognitive system* relevant to numbers, like:

- *parallel individuation*: an ability to pay attention to multiple things at once,
- *approximate number system*: in the most general lines it corresponds to an ability to estimate a cardinality of finite sets, or
- *natural-language quantification*: an ability to use some quantificational resources in the language (singular/plural, one/more than one, etc.).

According to cognitive scientists, these cognitive systems are not powerful enough or exact enough to represent natural numbers.

To the first fold corresponds also innate cognitive systems relative to computability.

4.2 Stage two: Computations at work

Cognitive scientists agree that for representing numbers, one need to be able to use language: *number words*, and *counting routine*. They also underline that a numerical system has to be able to support addition and multiplication.

It is not objective of this paper to confirm this interpretation of cognitive scientists research. The objective is to show conceptual coherence of this approach. It is hence claimed that in order to understand what are natural numbers one needs to know intuitively what “to compute” means. Proficiency in computing is claimed to have two stages:

- knowledge of the few initial *number words* and a *counting routine*: these two intuitions spelled out by cognitive scientists, correspond to the intuitions concerning the existence of zero and of a successor function;
- intuition of how to add, multiply and eventually to perform some other computable arithmetical functions: it does not have to be done at this stage with respect to the structure of all natural numbers, but intuitively understood on some small initial segment of them.

Psychologists claim that at this stage the concept of numbers is understood. However, the ambition of this paper is to extend this epistemological line to the concept of natural numbers from model-theoretical context.

4.3 Stage three: Descriptive definitions

In model-theoretical framework natural numbers are captured with descriptive formal definitions. In this paper we claim that natural numbers defined by PA1 with computability constraint on interpretation of function symbols. Alike, the concept of computability can be formalised in many ways, as a class of recursive functions or functions computable by Turing machines. Computability is here defined as Turing computability on strings of characters. In consequence, the presented conceptual analysis is in line with cognitive scientists investigations, discussed in the two previous stages, and also corresponds to the standpoint defended by computational structuralists. Additionally it shows how to coherently claim for intuitive computability of addition and multiplication, and preserve model-theoretic approach.

5 Bibliography

Carey (2009), The Origin of Concepts, OUP
Dehaene (1992), "Varieties of numerical abilities", Cognition, 44(1-2): 1-42
Dehaene (1999), Number Sense, OUP
Gilmore, Spelke (2008), "Children's understanding of the relationship between addition and subtraction", Cognition, 107(3): 932-945
Halbach, Horsten (2005), "Computational Structuralism", Philosophia Mathematica, 13(2): 174-186
Lemer, Dehaene, Spelke, Cohen (2003). "Approximate quantities and exact number words: dissociable systems", Neuropsych. 41(14): 1942–1958
Quinon, Zdanowski (2005), "Intended Model of Arithmetic: The Argument from Tennenbaum's Theorem", in: Cooper, et al. (eds.), Computation and Logic in the Real World, CiE 2007 Proceedings
Sarnecka, Lecture at University of California-Irvine,
http://ocw.uci.edu/lectures/lecture.aspx?id=315
Shapiro (1997), Structure and Ontology, OUP
Shapiro (2011), "Open-texture, computability and Church's thesis", talk at Trends in Logic IX
Spelke (2008), "Core Knowlegde and Cognitive Development", talk in College de France
Waismann (1951), "Verifiability", in: Ryle, Flew (eds.), Logic and Language, Blackwell
Wright (2000), "Neo-Fregean Foundations for Real Analysis", NDJFL, 41(4): 317-33