Visualization in topology: illustrations vs diagrams

1/ Diagrammatic reasoning: It is not what you see

It is a fact that visual representations of various nature are ubiquitous in mathematics as well as in many other human activities. These representations have been commonly labeled ‘visualization’.
This term is actually vague: it has been employed in many different contexts in order to refer to heterogenous processes. The starting point of this article is to clarify the connection between visualization and insight, understanding, and explanation in the mathematical practice.

Traditionally, the heuristic role of visualization has been accepted uncontroversially. Nevertheless, visualization has been often denied epistemic value. This assumption has been based on two dichotomies that we will try to reject: first, the dichotomy between two contexts in science, the one of discovery and the one of justification, and secondly the dichotomy between two kinds of reasoning, visual and linguistic reasoning.

In our scenario, the appropriate approach to visualization is the consideration of the way in which figures are introduced, interpreted and manipulated in mathematical practice. Each visualization is *prima facie* a figure, i.e. a physical object, drawn on a page or on a computer screen. A figure can be interpreted in various ways; just once interpreted in a context, it becomes an illustration or a diagram, that is, it is seen as a *representation*. Figures are always particular, but illustrations and diagrams must be generic, i.e. they are independent of some specificities of the drawing, for example they are considered up to the materiality or the location of the drawing itself. An illustration is static, while a diagram is dynamic. Diagrams in fact are not just depictions, but *dynamic* tools. Experts have acquired a *manipulative* imagination which allows them to take advantage of the multiple meanings which can be associated to diagrams. We will clarify and justify this framework presenting our case-study.

To carry on our analysis, we will refer a specific case-study: *Knot theory*. Through the specificity of this example we will examine the practice of mathematics and we will study diagrammatic reasoning form the ‘inside’ of an actual domain. Knot theory is a branch of topology which deals with mathematical knots, abstractions of physical knots (like the ones made with shoelaces). This field is suited because it requires the use of visual material of variegated kind and it involves the definition of diagrams endowed with different possible dynamics on. For this reason, it is an extraordinary rich source of examples.

2/ Discussion of knot theory

In the following, we arrange in a list the features of knot diagrams which are reveled by our analysis of knot theory.

(I) *A knot diagram emerges from a figure through interpretation*

A diagram can not have a meaning without interpretation: the meaning of a diagram is fixed by the context of use. We have to transcend the particular figure and get rid of many of its visual features, attending only to the relevant information.

(II) *A knot diagram incorporates the set of its possible moves*

The rules of motions for the space defined by knot diagrams are given in the interpretation step. The same figure can be interpreted as many different diagrams, it is only the definition of its possible manipulations that fixes its meaning. It is the practice - and therefore the chosen context - that solves the intrinsic ambiguity of a particular figure and discloses the diagram.

(III)*A knot diagram is a dynamic tool which triggers manipulative imagination*

Their inferential power lays on the number of different possible configurations and as a consequence on the degrees of freedom which allow the experts to move between possible manipulations. In this sense a diagram becomes a 'playground' for the mathematician.
The dynamic nature of knot diagrams calls for a form of manipulative imagination, which is to some extent a widening of our spatial perception as well as of our physical intuitions. However, this manipulative imagination, has to be trained by the specific practice of mathematics and it is an elaboration of our more precocious capacities such as vision or agency.

(IV)*Diagrams provide justification*

It is possible to demonstrate the existence of non-trivial knots relying abundantly on diagrams. This shows that knot diagrams, if situated in the right context and consistently with the shared practice of interpretation and manipulation, can be justificatory for our reasoning. The discard of the dichotomy vision versus language goes along with the discard of the one regarding the context of discovery versus justification.

3/ Plan of the article

In our article, we will first give a crash-course in knot theory and we will identify different representation 'levels' for its objects, having various (epistemic) roles and forms. Secondly, we will present a collection of examples pointing at the indispensability of dynamic diagrams in order to study knots. In the conclusion we will discuss these examples in the light of our considerations and value the possibility of extending our results to other domains of mathematics and human reasoning.

Our case-study allows us to analyze a diagrammatic practice. Knot diagrams are an example of diagrams on which we have some control, even if not entirely syntactic. The role of diagrams in the practice of knot theory is not reducible to the domain of pure heuristics but encompasses vast areas, such as the one of imagination, discovery and proving.

Our hypothesis is that such an operational framework for diagrams is not only valid for knot theory but can be generalized to other fields of mathematics and eventually to other forms of diagrammatic reasoning. Diagrams are not only visual prompts, but a way of externalizing thought and their functioning results from a complex synthesis of many different cognitive capacities, from more precocious to more sophisticated ones. Therefore, diagrams are a striking case for the interaction of nature and nurture in human reasoning.