Paradoxes of interacting modal predicates

Conceiving modal notions as predicates has been around since the very beginning of formal philosophizing. Carnap's "Die logische Syntax der Sprache" and Quine's "Three Grades of Modal Involvement" serve as but one example for such proposals. However, as has been pointed out, amongst others, by Montague when conceiving of modal notions as predicates we have to face paradoxical consequences similar to Tarski's indefinability theorem in the case of truth. That is, modal principles which are meant to be constitutive of the modal notions under consideration lead to inconsistency and thus our basic linguistic and philosophical intuitions with respect to these notions have to be reconsidered, if we wish to treat them as predicates.

Before doing so it seems helpful to systematize combinations of modal principle according to their joint consistency and inconsistency. To a certain extent systematization of this kind can be found in the seminal paper of Friedman and Sheard "An axiomatic approach to self-referential truth", but Egré's more recent "The Knower Paradox in the Light of Provability Interpretations of Modal Logic" focuses on a systematization of consistency and inconsistency results with respect to the underlying modal logic and is therefore more pertinent when it comes to examining theories of modal predicates with respect to their consistency or, respectively, their inconsistency. In our presentation we shall elaborate on Egré 's idea of using modal operator logics equipped with fixed-points constants (or, alternatively fixed-point operators) for classifying certain inconsistency results arising when modalities are treated as predicates.

Where Egré (and Friedman and Sheard) deal with the case of single modalities we shall be interested in the case of multiple modalities which are allowed to interact. Recently, Leitgeb and Horsten ("No Future") and Halbach ("On a side effect of solving Fitch's paradox through typing") have pointed out that further, unexpected inconsistencies might arise, if we allow for such multiple, interacting modal predicates. As valuable as these contribution are, they leave open whether genuinely new paradoxes arise in this setting. That is to say, it is an open question whether the inconsistencies arise because of the very interaction of the modal predicates or because of the underlying modal logic of one of the modal notions has been strengthened to the effect that inconsistency arises along the lines of the well known paradoxes of single modalities.

Where this distinction is admittedly vague and imprecise we shall try to elaborate on this idea and propose one explication of it which we specify using modal logics with fixed-point constants. Roughly, the idea is as follows:

Let $\mathcal{L}_\Box$ and $\mathcal{L}_\blacksquare$ be two propositional modal languages each containing a one-place modal operator, and $\mathcal{L}_{\Box\blacksquare}$ the propositional modal language containing both modal operators. Moreover, let $\delta_{\phi(p)}$ be a fixed-point of $\phi(p)$ and set a fixed-point axiom for $\delta_{\phi(p)}$ to be the formula

$$\phi(\delta_{\phi(p)}) \leftrightarrow \delta_{\phi(p)}$$

Now, for a modal logic S, we denote S^F to be the extension of S by the fixed-point axioms for the fixed-point constants of the language under consideration. We then say an inconsistency to be genuine (i.e. not reducible to one of the paradoxes of single modal notions) if a modal logic

S^F is inconsistent in $\mathcal{L}_{\Box\blacksquare}$ with fixed-point for all modal formulas of the language but consistent in $\mathcal{L}_{\Box\blacksquare}$ when only fixed-point constants for modal formulas of $\mathcal{L}_{\Box}$ and $\mathcal{L}_{\blacksquare}$ are added to the language.

Applying this analysis we can show the paradox presented by Horsten and Leitgeb to be genuine, that is to require a fixed-point for a formula being a formula of $\mathcal{L}_{\Box\blacksquare}$ but not of $\mathcal{L}_{\Box}$ nor of $\mathcal{L}_{\blacksquare}$, e.g. the fixed-point the fixed-point for the formula $\neg\Box\blacksquare p$, i.e. $\delta_{\neg\Box\blacksquare p}$ will do.

On the other hand, assuming very basic closure principles for one of the modalities involved, we can show Halbach's paradox not to be genuine, that is, it is reducible to well known paradoxes, as it can be derived using a fixed-point of a formula of $\mathcal{L}_{\Box}$, e.g. $\delta_{\neg\Box p}$.