A DEFENSE AND AN IMPROVEMENT OF DIODORUS-PRIOR'S PROOF

by

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Abstract. — This paper is a reply from an intuitionistic point of view to three criticisms made by Vuillemin [[**4**], pp. 8-14] against Prior's logical reconstruction of Diodorus' Master Argument (for short "Diodorus-Prior's proof"). Indeed, the intuitionistic interpretation of Prior's proof can rule out the first two criticisms and limit the scope of the third. Last, we prove that the conclusion of Diodorus' Master Argument can be deduced from only one undischarged assumption, i.e. premise (A). This new proof is therefore an improvement of Didorus-Prior's, because it shows clearly that premise (A) is incompatible with premise (C).

1. Prior's Proof of Diodorus' Master Argument

It is well known that Prior [**3**] has given a very concise and elegant proof of famous Diodorus' Master Argument (for short MA). The object language in which we are going to translate Prior's proof is standard for a temporal modal logic; it is the set of symbols

$$\{p, q, r, \bot, \neg, \wedge, \vee, \rightarrow, \Diamond, \Box, P, F\}$$

which are respectively propositional variables p, q, r, falsity constant, negation, conjunction, disjunction, conditional, modal operator of possibility, modal operator of necessity, temporal operators to express that p was the case (Pp) or that q will be the case (Fq).

According to Epictetus, MA was based on the following three premises.

- Every true proposition concerning the past is necessary, i.e. whatever has been the case cannot now not have been the case; in formula:
$$Pp \to \neg\Diamond\neg Pp \tag{A}$$
- The impossible does not follow from the possible, i.e. If p necessarily implies q, then if q is not possible, p is not possible, i.e.
$$\Box(p \to q) \to (\neg\Diamond q \to \neg\Diamond p) \tag{B}$$
- Something that neither is nor will be is possible, i.e. there is a possible p which neither is presently true nor will be so, i.e.
$$\Diamond p \wedge \neg p \wedge \neg Fp \tag{C}$$

Diodorus claims that the assumption of the conjunction of (A) and (B) implies the negation of (C) i.e. if some event p neither is the case nor will ever be the case, then p is not possible:
$$(\neg p \wedge \neg Fp) \to \neg\Diamond p \tag{$\neg$ C}$$

In order to give a formal proof of MA, Prior makes use of two additional premises. The first says that from a thing's being the case, it necessarily follows that it has always been going to be the case. It is an axiom of the minimal temporal logic $\mathbf{K_t}$, that I write here with label (HF) like Garson [**1**] :
$$\Box(p \to HFp) \tag{HF}$$

But Prior prefers to express the operator H by its *definiens*, i.e. by $\neg P \neg$ and therefore chooses as the first premise the following formula:
$$\Box(p \to \neg P \neg Fp) \tag{D}$$
which means that, necessarily if something is the case, then it has never-been never-going-to-be the case.

The second premise added by Prior is
$$(\neg p \wedge \neg Fp) \to P\neg Fp \tag{E}$$
i.e. if p is not the case and never will be so, then it has been the case that p will never be the case.

Then, it is possible to prove in natural deduction that $\neg C$ is a logical consequence of the set of premises (A), (B), (D) and (E), i.e.
$$(A), (B), (D), (E) \vdash (\neg C)$$
.

Proof. —

1	$Pp \to \neg\Diamond\neg Pp$	(A)
2	$\Box(p \to q) \to (\neg\Diamond q \to \neg\Diamond p)$	(B)
3	$\Box(p \to \neg P\neg Fp)$	(D)
4	$(\neg p \wedge \neg Fp) \to P\neg Fp$	(E)
5	$\quad \mid\ \neg p \wedge \neg Fp$	**H**
6	$\quad \mid\ P\neg Fp$	$\to E$, 5,4
7	$\quad \mid\ P\neg Fp \to \neg\Diamond\neg P\neg Fp$	$p/\neg Fp$, 1
8	$\quad \mid\ \neg\Diamond\neg P\neg Fp$	$\to E$, 6,7
9	$\quad \mid\ \Box(p \to \neg P\neg Fp) \to (\neg\Diamond\neg P\neg Fp \to \neg\Diamond p)$	$q/\neg P\neg Fp$, 2
10	$\quad \mid\ \neg\Diamond\neg P\neg Fp \to \neg\Diamond p$	$\to E$, 3, 9
11	$\quad \mid\ \neg\Diamond p$	$\to E$, 8, 10
12	$(\neg p \wedge \neg Fp) \to \neg\Diamond p$	$\to I$, 5, 11.

□

2. Vuillemin's criticisms against Prior's proof

2.1. First objection. — Vuillemin [[**4**], pp. 7-8] wonders about the signification of premise (A). First, Vuillemin stresses that (A) claims that every true statement about the past is necessarily true and that such a reading of (A) prevents to understand (A) as claiming that every statement grammatically in the past is necessarily true.

Second, Vuillemin holds that "each axiom of Epictetus requires an interpretation in terms of temporal modalities", because in his opinion "the logical sense of modalities, as such, excludes times". The following quotation shows that Vuillemin has in mind an Aristotelian reading of (A):

> Supposing it is necessary that a certain thing should have happened, it is *a fortiori* possible that that thing should have happened. But when modalities are applied to temporal events, they are generally understood, and rightly so, in a different sense. They are taken in real, rather than in a logical sense. Irrevocability, which is a factual kind of necessity, applies to any event whatsoever, even a contingent one, once it has come to pass. It follows that the real modality itself must be assigned a temporal index distinct from the one affecting the event to which the modality applies. At present

> it is irrevocable, or necessary in the factual sense, that the battle of Salamis took place. Factual possibility, the modal counterpart of this factual necessity, will likewise be assigned a temporal index of its own. But it is notable that there is no way of getting from the past conceived as a factual necessity to the corresponding factual possibility, where that factuality is taken to be that of a future or, at most, a present event, to the exclusion of any event having taken place. We shall see that for Aristotle this priviliged temporal direction constitutes the entire content of the Master Argument's first premise.

Thus, Vuillemin believes that the only way to avoid the ambiguity of premise (A) as well as the whole of MA is to apply temporal indexes to modalities; according to Vuillemin, MA makes sense vis-à-vis the real modalities and not vis-a-vis their logical sense. This is the first criticism made by Vuillemin against Prior's proof of MA.

2.2. Second objection. — Vuillemin claims that first additional premise (D) involves a principle of retrogradation of truth which is explicitly rejected by Aristotle (see *De Int.*, 9,18$^{\mathrm{b}}$34). MA is clearly anti-Aristotelian, therefore Prior made a mistake by using a premise which is clearly rejected by Aristotle.

2.3. Third objection. — The third and last criticism made by Vuillemin against Prior's proof, is that premise (E) is valid only if time is discrete, and therefore this additional premise is "in direct opposition to the Aristotelian theory of the continuity of time in book V of the *Physics*." That is why the conclusion made by Vuillemin [[**4**], p. 12] about Prior's proof is somewhat severe:

> So, Diodorus' argument, on Prior's reconstruction, would fail to touch Aristotle in the least. Not only would it not touch him in fact, but the introduction of clause (E) of the discontinuity of time would be tantamount to a conscious admission of defeat.

3. Intuitionistic replies to Vuillemin's objections

3.1. Reply to the first objection. — Vuillemin could have made the following stronger criticism of axiom (A). Note that axiom (A) has the logical form of the conditional necessity (for short CN; CN is defined in the next section of this paper):

$$p \to \neg\Diamond\neg p \tag{CN}$$

and that (CN) is in *classical* modal logic equivalent to

$$p \to \Box p \tag{Nec}$$

It is not difficult to prove that Prior's proof requires at least the modal system **T** which is basically defined by

$$\Box p \to p \tag{T}$$

But the acceptance of (Nec) and (T) entails that the equivalence

$$p \leftrightarrow \Box p \tag{T\&Nec}$$

would be involved by Prior's proof. If there is no way to avoid such a result, it means that Prior's proof would collapse into the *Trivial system* (**Triv.** for short) i.e. into a system which expresses only the tautologies of Propositional Calculus and where $\Box$ and $\Diamond$ are in fact useless symbols. Indeed, Hugues and Cresswell [[**2**], p. 65] have clearly explained that the add of formula (T&Nec) to the weakest normal modal system, i.e. **K**, entails the system **Triv**.

But there is a reasonable way to avoid such a disaster: in intuitionistic modal logic the move from (CN) to (Nec) is blocked. Indeed, in intuitionistic modal logic it is provable that

$$\nvdash_i (p \to \neg\Diamond\neg p) \to (p \to \Box p) \tag{1}$$

and (1) means that (CN) is intuitionistically *weaker* than (Nec). The intuitionistic reading of (CN) is: "if it is proved that p is the case, then it is also proved that it is not possible to prove that p is logically refutable, i.e. it is not possible to prove that p entails a contradiction". (Nec) says more: "if it is proved that p is the case, then it is proved that *necessarily* p is the case." Therefore we suggest that the intuitionistic use of $\neg\Diamond\neg p$ is sufficient to distinguish what can be understood as the irrevovability of the past and the logical necessity of constructive mathematics. In intuitionistic modal logic (A) means precisely that if p has been the case, then it is not possible that it has never been the case. This intuitionistic reading of (A) means that the acceptance of

$$Pp \to \neg\Diamond\neg Pp \tag{A}$$

does not entail the acceptance of its classical equivalent

$$Pp \to \Box Pp \tag{NecP}$$

the latter being of course more metaphysically loaded.

3.2. Reply to the second objection. — Understood from the point of view of intuitionistic logic, the additional premise (D) cannot be charged of being guilty of any retrogradation of truth. Indeed,

$$\Box(p \to \neg P\neg Fp) \tag{D}$$

means that, necessarily if p is the case, then the statement of the existence in the past of a proof that p will never be refuted. Clearly there is only a retrogradation of *refutation*, and not a retrogradation of proof of positive statements. Retrogradation of refutation is involved in the meaning of all intuitionistic tautologies with negative consequents. For example

$$\vdash_i p \to \neg\neg p \tag{2}$$

means that if p is the case, or if one gets a proof of p, then the refutation of p is itself refuted. That is why you may read in Logic textbooks that proofs of positive statements "prove forward", while proofs of negative statements "prove backward".

Last but not least, the formula that (D) claims to be necessary, i.e.

$$p \to \neg P \neg F p$$

is a theorem of $\mathbf{K_t}$, i.e. a theorem of minimal temporal logic. It seems theredifficult from a logical point of view to reject (D).

3.3. Reply to the third objection. — It is right that (E) is provable in a system $\mathbf{S}$ of temporal logic only if one admits in $\mathbf{S}$ that the structure of time is discrete. In minimal temporal logic $\mathbf{K_t}$ the additional premise (E) is unprovable. In *The topology of time*, Prior [[**3**], p.49] shows his doubts about premise (E):

> Just this Proposition 5 [i.e. (E)], however, had begun about 1960 to strike me as dubious. Theses which appeal, in order to gain intuitive plausibility, to what was the case at the 'moment just past', are liable to commit one to the view that time is discrete. What if there is *no* 'moment just past', but between any past moment, however close to the present, and the present itself, there is another moment still past? On this supposition, Proposition 5 in fact fails.

If Prior himself has raised doubts about (E), Vuillemin seems to be justified in rejecting the use of this premise because it involves the discreteness of time, a thesis rejected by Aristotle. Nevertheless, we are going to see that there are reasons for removing Prior's doubts and counteracting Vuillemin's third objection.

Let us remind that (E) says that if one supposes an event p which neither is nor ever will be the case, then one must accept the existence of one past moment where p will never be the case. First, there is one counter-model showing that this formula is not provable in all structures of time: suppose that $\neg p$ is true at the first moment of time, i.e. that there is no past time before the moment where $\neg p$ is true, then (E) is clearly false. But if one refuses to assume the existence of a first moment of time, this counter-model can be rejected.

Second, because of the continuous structure of time, it is not possible to *decide* the instant t_n where p is the case in the past 'just before' the moment where p neither is nor will ever be the case. Now, take for example Socrates' death. At the moment of his death, neither is the case that Socrates is alive, nor it will ever be the case that he is alive, and necessarily, it must exist one instant in the past where it is true that Socrates will never be alive, because the refusal of this conclusion leads to the acceptance of Zeno's paradoxes. Let us remind that Aristotle rejected these paradoxes for being based on the false assumption that it is impossible for a thing to pass over infinite things in a finite time. According to Aristotle, Zeno's did not see that one can conceive any finite moment as composed by a potentially infinite number of *instants*:

> [...] to the question whether it is possible to pass through an infinite number of units either of time or of distance we must reply that in a sense it is and in a sense it is not. If the units are actual, it is not possible: if they are potential, it is possible. (*Physics* V 263^{b}2-5).

Thus, in arguing on behalf of Aristotle that premise (E) is not admissible from Aristotle's point of view, Vuillemin commits a fallacy. He forgets that for Aristotle time is continuous only because every unit of time, as small as it can be, is itself *potentially* infinite. This means that every finite movement can be always conceived of as passing through a finite numbers of units of time. In other words, every portion of time contains an infinite number of potential divisions, but every development of sublunar phenomena can be understood only via a finite number of finite moments. Vuillemin forgets also that for Aristotle the concept of potentiality is meaningful for the future, not for the past. Therefore there are good historical and philosophical reasons to reject Vuillemin's third objection against Prior's proof of MA.

Last, the fact that (E) is not provable in minimal temporal logic is not, in our opinion, a serious problem. (E) is *satisfiable* in $\mathbf{K_t}$, and that is enough to claim that (E) is a non-logical axiom useful in Prior's proof of MA.

4. Improving the proof of Diodorus' Master Argument

In the last paper devoted to MA, Vuillemin [**5**] proposes the correction of an error that he made in *Nécessité et Contingence* [**6**] in the proof of MA, and before the development of his new formal proof of MA, he says very clearly:

> The conjunction of (A), (B), (C) and of the conditional necessity entails a logical impossibility. Therefore this conjunction is not logically possible, Q.E.D.

The definition of conditional necessity (for short CN) can be expressed in the following quantified form:

> For any time t, if p takes place during time t, it is necessary during time t that p take place during time t.

The formal expression of CN is

$$p_t \to \neg\Diamond_t\neg p_t \tag{CN}$$

It is good thing to prove that a conjunction of propositions entails a contradiction; but it is better to know which proposition in this conjunction must be considered as false. In this last section, we are going to prove that (A) is a *sufficient condition* to deduce the negation of (C). In other words, there is no need of (B), neither (D) nor (E) to prove from (A) the negation of (C). The following proof is both improves Prior's proof and agrees with Vuillemin's philosophical analysis: the assumption of CN is crucial to understand Diodorus' conclusion. Four basic points before giving our proof:

1. There are good reasons to see (A) as an expression of CN: if one gets a proof of a proposition Pp, i.e. a proposition about a past event, then it is absurd to claim the possibility of refuting Pp because if Pp is true or proved, this proposition about this past event is true and will be true for ever. Of course for the same reasons if one replace p by $\neg p$ in (A), this negative version of (A) remains an expression of CN.
2. If one accepts the convention according to which, when p is *not* in the scope of a temporal operator, p and $\neg p$ mean respectively: 'now p is the case' and 'now, p is not the case', then the implications $p \to \neg\Diamond\neg p$ and $\neg p \to \neg\Diamond\neg\neg p$ are also admissible expressions of CN. Intuitively, if its proved that *now* p is the case, then it is absurd to claim that p is refuted. For example, if I know that I am thinking, then I know that it is absurd to say that I am not thinking. In the same way, if there is evidence that *now* p is not the case, for example that I am not standing in front of the door, then the refutation of $\neg p$ is impossible. If it is true that *now* I am not standing in front of the door, it is impossible to change at the same time the content of this 'now', just as standing and sitting at the same instant is impossible.
3. It is well known that the assumption of the contingency of p means by definition $\neg\Box p \wedge \neg\Box\neg p$. In the following proof, we make an implicit but correct use of the rule of conjunction elimination: only $\neg\Box\neg p$ is supposed in this proof.
4. Last, a couple of logical equivalences provable in intuitionistic modal logic are used in this proof, i.e. $\neg\Diamond\neg\neg p \leftrightarrow \Box\neg p$ and $\neg\neg\Box\neg p \leftrightarrow \neg\Diamond p$.

It is now possible to prove intuitionistically the following formal inference:

$$Pp \to \neg\Diamond\neg Pp \vdash_i (\neg p \wedge \neg Fp) \to \neg\Diamond p$$

Proof. —

1	$Pp \to \neg\Diamond\neg Pp$	(A)
2	$\quad\mid\ \neg p \wedge \neg Fp$	**H**
3	$\quad\mid\quad\mid\ \neg\Box\neg p$	**H**
4	$\quad\mid\quad\mid\ \neg p$	$\wedge_l E$, 2
5	$\quad\mid\quad\mid\ \neg p \to \neg\Diamond\neg\neg p$	$Pp/\neg p$, 1
6	$\quad\mid\quad\mid\ \neg\Diamond\neg\neg p$	$\to E$, 4,5
7	$\quad\mid\quad\mid\ \Box\neg p$	$6 \leftrightarrow 7$
8	$\quad\mid\quad\mid\ \bot$	$\neg E$ 7, 3
9	$\quad\mid\ \neg\neg\Box\neg p$	$\neg I$, 3-8
10	$\quad\mid\ \neg\Diamond p$	$9 \leftrightarrow 10$
11	$(\neg p \wedge \neg Fp) \to \neg\Diamond p$	$\to I$, 2-10

□

This proof shows clearly that all philosophical disputes about (B) (D) and (E) can be avoided in order to focus only on the significance of conditional necessity i.e. premise (A).

References

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[6] Vuillemin, J. *Nécessité ou contingence: l'aporie de Diodore et les systèmes philosophiques.* Ed. de Minuit, Paris, 1997.

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