Abstraction Principles: a Modal Interpretation

Abstraction Principles (APs), initially used by Frege (1879, 1884, 1892, 1903) in his system of foundations of mathematics, are recently being re-introduced into foundational studies within a fairly new approach called neologicism (Wright 1983; Zalta 1983; Hale and Wright 2001). APs are expressions of the form: '$f(x) = f(y) \equiv xRy$', where R is an equivalence relation and f is a newly introduced operator. Roughly speaking, on the neologicist's view, an AP is ideally meant to fix reference of abstract terms and explicate operation f which on the intended interpretation assigns abstract objects to objects that already belong to the domain, or to concepts reaching over that domain. For instance, Hume's Principle says that the number of one concept is the same as the number of another concept if and only if those concepts are equinumerous (a notion defined independently of the notion of number). Thus, by introducing this principle, we're supposed to: fix reference of expressions like 'the number of F', determine an operation that assigns numbers to objects, and explicate our sortal concept of a number. In fact, adding comprehension principle for concepts and Hume's Principle to second-order logic allows to derive second-order Peano Arithmetic.

Difficulties that neologicism runs into are fairly well-known (Fine 2002). The neologicist would like to claim that those APs which they found suitable are analytically (or at least conceptually) true. But giving a rationale for such a claim is far from trivial. One can't say that all APs are true, because (within a sensible logical framework) certain APs lead to straightforward contradictions. One can't even require that all consistent APs are true, because certain APs are separately consistent but mutually exclusive. Interestingly, there also is a set of APs such that no finite conjunction of them is inconsistent, and yet they all cannot be satisfied in a domain. These and related difficulties give rise to the fairly open problem of finding sensible acceptability conditions of APs. All suggestions put forward so far are quite complicated and there is no general agreement as to their plausibility and effectiveness.

More philosophical issues have also been raised. It is doubtful that APs are capable of *fixing* reference of abstract terms. Given a domain of non-abstract objects, no AP determines unambiguously the set of abstract objects that have to be added to this domain in order for the principle to hold. Even if we restrict ourselves to a certain cardinality of such an extended model, certain principles will still have non-isomorphic models and all APs will be insensitive to permutations of abstract objects. Also, APs have been claimed to be unable to provide truth-conditions of "mixed" identity statements like 'the number of F = Julius Caesar'.

I would like to consider an alternative approach to APs, inspired by Kotarbiński (1929), a Polish logician and Tarski's teacher. On his view, there are no abstract objects and the only reason we introduce terms that seemingly refer to them is brevity and simplicity of our discourse. Thus, (on philosophical grounds) he divides singular terms into those that really refer to objects and those that only pretend to do that and behave like singular terms, but in fact don't refer to anything. The latter he dubs *onomatoids*.

The anti-Fregean diagnosis of the problems encountered by neologicism, as I would construe it, is this: it's not APs that cause the problem. It's the belief that they establish functions *into* the domain, that is, that abstract terms introduced by means of APs really refer to objects. Once we treat APs as *linguistic rules* that tell us what onomatoids can be introduced and what differences between them we are to ignore to increase simplicity of our discourse, most of the neologicist's problems disappear, and yet, we can still get our mathematical theories to behave the way we want them to behave. I will elaborate on this claim using the example of Hume's Principle and constructing a formal theory that handles introduction of onomatoids by means of abstraction principles. Given this framework, I will explain how non-trivial truth conditions for propositions of arithmetic can be given, even in the absence of numbers. I will also argue that this approach can deal with the problems encountered by neologicism.

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